

Entanglement robustness and geometry in systems of identical particles

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Abstract

The robustness properties of bipartite entanglement in systems of N bosons distributed in M different modes are analyzed using a definition of separability based on commuting algebras of observables, a natural choice when dealing with identical particles. Within this framework, expressions for the robustness and generalized robustness of entanglement can be explicitly given for large classes of boson states: their entanglement content results in general much more stable than that of distinguishable particles states. Using these results, the geometrical structure of the space of N boson states can be explicitly addressed.

1 Introduction

When dealing with many-body systems made of identical particles, the usual definitions of separability and entanglement appear problematic since the natural particle tensor product structure on which these notions are based is no longer available. This comes from the fact that in such systems the microscopic constituents can not be singly addressed, nor their properties directly measured [1, 2].

This observation points to the need of generalized notions of separability and entanglement not explicitly referring to the set of system states, or more in general to the “particle” aspect of first quantization: they should rather be based on the second quantized description of many-body systems, in terms of the algebra of observables and to the behavior of the associated correlation functions. This “dual” point of view stems from the fact that in systems of identical particles there is not a preferred notion of separability, it can be meaningful only in reference to a given choice of (commuting) sets of observables [3]-[8].

This new approach to separability and entanglement have been formalized in [9]-[12]: it is valid in all situations, while reducing to the standard one for systems of distinguishable particles;¹ in particular, it has been successfully applied to the treatment of the behavior of trapped ultracold bosonic gases, giving rise to new, testable prediction in quantum metrology [9]-[11].

As in the case of systems of distinguishable particles,² suitable criteria able to detect non-classical correlations through the implementations of practical tests are needed in order to easily identify entangled bosonic many-body states [28]-[30]. In the case of bipartite entanglement, the operation of partial transposition [31]-[34] turns out to be again very useful; actually, it has been found that in general this operation gives rise to a much more exhaustive criterion for detecting bipartite entanglement than in the case of distinguishable particles [11, 12]. As a byproduct, this allows a rather complete classification of the structure of bipartite entangled states in systems composed of N bosons that can occupy M different modes [12], becoming completely exhaustive in some relevant special cases, as for $M = 2$.

In the following we shall further explore the properties of bipartite entanglement in bosonic systems made of a fixed number of elementary constituents. We shall first study to what extent an entangled bosonic state results robust against its mixing with another state (separable or not): we shall find that in general, bosonic entanglement is much more robust than the one of distinguishable particles. In particular, we shall give an explicit expression for the so-called “robustness” [35] and upper bounds for the “generalized robustness” [36]. As a byproduct, a characterization of the geometry of the space of bosonic states will also be given; the structure of this space results much richer than in the case of systems with N distinguishable constituents. One of the most striking results is that the totally mixed separable state, proportional to the unit matrix, does no longer lay in the interior of the subspace of separable states: in any of its neighborhood entangled states can always be found.

2 Entanglement of multimode boson systems

As mentioned above, we shall focus on bosonic many-body systems made of N elementary constituents that can occupy M different modes. From the physical point of view, this is a quite general framework, relevant in the study of quite different systems in quantum optics, atom and condensed matter physics. For instance, this theoretical paradigm is of special importance in modelling the behavior of ultracold bosonic gases confined in multi-site optical lattices, that are becoming so relevant in the study of quantum phase transitions and other quantum many-body effects (*e.g.*, see [25, 26], [37]-[44] and references therein).

In order to properly describe the N boson system, let us thus introduce creation a_i^\dagger and

¹The notion of entanglement in many-body systems has been widely discussed in the recent literature (*e.g.*, see [13]-[24]); nevertheless, we stress that only a limited part of those results are relevant for identical particles systems.

²For general reviews on the role of quantum correlations in systems with large number of constituents see [25]-[27].

annihilation operators a_i , $i = 1, 2, \dots, M$, for the M different modes that the bosons can occupy, obeying the standard canonical commutation relations, $[a_i, a_j^\dagger] = \delta_{ij}$. The total Hilbert space \mathcal{H} of the system is then spanned by the many-body Fock states, obtained by applying creation operators to the vacuum:

$$|n_1, n_2, \dots, n_M\rangle = \frac{1}{\sqrt{n_1! n_2! \dots n_M!}} (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} \dots (a_M^\dagger)^{n_M} |0\rangle, \quad (1)$$

the integers n_1, n_2, \dots, n_M representing the occupation numbers of the different modes. Since the number of bosons is fixed, the total number operator $\sum_{i=1}^M a_i^\dagger a_i$ is a conserved quantity and the occupation numbers must satisfy the additional constraint $\sum_{i=1}^M n_i = N$; in other words, all states must contain exactly N particles. As a consequence, the dimension D of the system Hilbert space \mathcal{H} is finite; one easily finds: $D = \binom{N+M-1}{N}$.

In addition, the set of polynomials in all creation and annihilation operators, $\{a_i^\dagger, a_i\}$, $i = 1, 2, \dots, M$, form an algebra that, together with its norm-closure, coincides with the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators;³ the observables of the systems are part of this algebra.

When dealing with systems of identical particles, instead of focusing on partitions of the Hilbert space \mathcal{H} , it seems natural to define the notion of bipartite entanglement by the presence of non-classical correlations among observables belonging to two commuting subalgebras \mathcal{A}_1 and \mathcal{A}_2 of $\mathcal{B}(\mathcal{H})$ [9]. Quite in general, one can then introduce the following definition:

Definition 1. *An algebraic bipartition of the algebra $\mathcal{B}(\mathcal{H})$ is any pair $(\mathcal{A}_1, \mathcal{A}_2)$ of commuting subalgebras of $\mathcal{B}(\mathcal{H})$, $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{B}(\mathcal{H})$.*

More explicitly, a bipartition of the M -oscillator algebra $\mathcal{B}(\mathcal{H})$ can be given by splitting the collection of creation and annihilation operators into two disjoint sets, $\{a_i^\dagger, a_i \mid i = 1, 2, \dots, m\}$ and $\{a_j^\dagger, a_j \mid j = m+1, m+2, \dots, M\}$; it is thus uniquely determined by the choice of the integer m , with $0 \leq m \leq M$. All polynomials in the first set (together with their norm-closures) form a subalgebra \mathcal{A}_1 , while the remaining set analogously generates a subalgebra \mathcal{A}_2 . Since operators pertaining to different modes commute, one sees that any element of the subalgebra \mathcal{A}_1 commutes with each element of \mathcal{A}_2 , in short $[\mathcal{A}_1, \mathcal{A}_2] = 0$.

Remark 1: *i)* Note that there is no loss of generality in assuming the modes forming the subalgebras \mathcal{A}_1 (and \mathcal{A}_2) to be contiguous; if in the chosen bipartition this is not the case, one can always re-label the modes in such a way to achieve this convenient ordering.

ii) Further, when the two commuting algebras \mathcal{A}_1 and \mathcal{A}_2 are generated only by a subset $M' < M$ of modes, one can simply proceed as if the N boson system would contain just the used M' modes, since all operators in $\mathcal{B}(\mathcal{H})$ pertaining to the modes not involved in the bipartition commute with any element of the two subalgebras \mathcal{A}_1 and \mathcal{A}_2 , and therefore effectively act as “spectators”. As a consequence, all the considerations and results discussed

³The algebra $\mathcal{B}(\mathcal{H})$ is generated by the so-called Weyl operators; all polynomials in the in the creation and annihilation operators are obtained from them by proper differentiation [45, 46].

below holds also in this situation, provided one replaces the total number of modes M with M' , the actual number of modes used in the chosen bipartition. \square

Having introduced the notion of algebraic bipartition $(\mathcal{A}_1, \mathcal{A}_2)$ of the system operator algebra $\mathcal{B}(\mathcal{H})$, one can now define the notion of local observable:

Definition 2. *An element (operator) of $\mathcal{B}(\mathcal{H})$ is said to be $(\mathcal{A}_1, \mathcal{A}_2)$ -local, i.e. local with respect to a given bipartition $(\mathcal{A}_1, \mathcal{A}_2)$, if it is the product $A_1 A_2$ of an element A_1 of \mathcal{A}_1 and another A_2 in \mathcal{A}_2 .*

From this notion of operator locality, a natural definition of state separability and entanglement follows [9]:

Definition 3. *A state ρ (density matrix) will be called **separable** with respect to the bipartition $(\mathcal{A}_1, \mathcal{A}_2)$, in short $(\mathcal{A}_1, \mathcal{A}_2)$ -separable, if the expectation $\text{Tr}[\rho A_1 A_2]$ of any local operator $A_1 A_2$ can be decomposed into a linear convex combination of products of expectations:*

$$\text{Tr}[\rho A_1 A_2] = \sum_k \lambda_k \text{Tr}[\rho_k^{(1)} A_1] \text{Tr}[\rho_k^{(2)} A_2] , \quad \lambda_k \geq 0 , \quad \sum_k \lambda_k = 1 , \quad (2)$$

where $\{\rho_k^{(1)}\}$ and $\{\rho_k^{(2)}\}$ are collections of admissible states for the systems; otherwise the state ρ is said to be **entangled** with respect the bipartition $(\mathcal{A}_1, \mathcal{A}_2)$.

Remark 2: i) This generalized definition of separability can easily be extended to the case of more than two partitions; for instance, in the case of an n -partition, Eq.(2) above would extend to:

$$\text{Tr}[\rho A_1 A_2 \cdots A_n] = \sum_k \lambda_k \text{Tr}[\rho_k^{(1)} A_1] \text{Tr}[\rho_k^{(2)} A_2] \cdots \text{Tr}[\rho_k^{(n)} A_n] , \quad \lambda_k \geq 0 , \quad \sum_k \lambda_k = 1 . \quad (3)$$

ii) When dealing with systems of *distinguishable* particles, one finds that *Definition 3* gives the standard notion of separability [9].

iii) In this respect, it should be noticed that when dealing with systems of identical particles, there is no *a priori* given, natural partition, so that questions about entanglement and separability, non-locality and locality are meaningful only with reference to a specific choice of commuting algebraic sets of observables [3]-[12]; this general observation, often overlooked in the literature, is at the basis of the definitions given in (2) and (3).

iv) A special situations is represented by pure states [12]. In fact, when dealing with pure states, instead of general statistical mixtures, and bipartitions that involve the whole algebra $\mathcal{B}(\mathcal{H})$, the separability condition in (2) (and similarly for (3)) simplify, becoming:

$$\text{Tr}[\rho A_1 A_2] = \text{Tr}[\rho^{(1)} A_1] \text{Tr}[\rho^{(2)} A_2] , \quad \rho = |\psi\rangle\langle\psi| , \quad (4)$$

with $\rho^{(1)}$, $\rho^{(2)}$ projections on the restrictions of $|\psi\rangle$ to the first, respectively second, partition; in other terms, separable, pure states turn out to be just product states. \square

Examples of N bosons pure separable states are the Fock states. Using the notation and specifications introduced before (*cf.* (1)), they can be recast in the form:

$$|k_1, \dots, k_m; k_{m+1}, \dots, k_M\rangle, \quad \sum_{i=1}^m k_i = k, \quad \sum_{j=m+1}^M k_j = N - k, \quad 0 \leq k \leq N, \quad (5)$$

where k indicates the number of bosons in the first partition; by varying it together with the integers k_i , these states generate the whole Hilbert space \mathcal{H} . This basis states can be relabeled in a different, more convenient way as:

$$|k, \sigma; N - k, \sigma'\rangle, \quad \sigma = 1, 2, \dots, \binom{k + m - 1}{k}, \quad \sigma' = 1, 2, \dots, \binom{N - k + M - m - 1}{N - k}, \quad (6)$$

where, as before, the integer k represents the number of particles found in the first m modes, while σ counts the different ways in which those particles can fill those modes; similarly, σ' labels the ways in which the remaining $N - k$ particles can occupy the other $M - m$ modes.⁴ In this new labelling, the property of orthonormality of the states in (5) becomes: $\langle k, \sigma; N - k, \sigma' | l, \tau; N - l, \tau' \rangle = \delta_{kl} \delta_{\sigma\tau} \delta_{\sigma'\tau'}$.

For fixed k , the basis vectors $\{|k, \sigma; N - k, \sigma'\rangle\}$ span a subspace \mathcal{H}_k of dimension $D_k D_{N-k}$, where for later convenience we have defined (*cf.* (6) above):

$$D_k \equiv \binom{k + m - 1}{k}, \quad D_{N-k} \equiv \binom{N - k + M - m - 1}{N - k}. \quad (7)$$

Remark 3: Note that the space \mathcal{H}_k is naturally isomorphic to the tensor product space $\mathbb{C}^{D_k} \otimes \mathbb{C}^{D_{N-k}}$; through this isomorphism, the states $|k, \sigma; N - k, \sigma'\rangle$ can then be identified with the corresponding basis states of the form $|k, \sigma\rangle \otimes |N - k, \sigma'\rangle$. \square

By summing over all values of k , thanks to a known binomial summation formula [47], one recovers the dimension D of the whole Hilbert space \mathcal{H} :

$$\sum_{k=0}^N D_k D_{N-k} = D = \binom{N + M - 1}{N}. \quad (8)$$

Using this notation, a generic mixed state ρ can then be written as:

$$\rho = \sum_{k,l=0}^N \sum_{\sigma, \sigma', \tau, \tau'} \rho_{k\sigma\sigma', l\tau\tau'} |k, \sigma; N - k, \sigma'\rangle \langle l, \tau; N - l, \tau'|, \quad \sum_{k=0}^N \sum_{\sigma, \sigma'} \rho_{k\sigma\sigma', k\sigma\sigma'} = 1. \quad (9)$$

In general, to determine whether a given density matrix ρ can be written in separable form is a hard task and one is forced to rely on suitable separability tests. One of the most useful

⁴Clearly, we need two extra labels σ and σ' for each value of k , so that these labels (as well the range of values they take) are in general k -dependent: in order to keep the notation as a simple as possible, in the following these dependences will be tacitly understood.

such criteria involves the operation of partial transposition [31, 32]: a state ρ for which the partially transposed density matrix $\tilde{\rho}$ is no longer positive is surely entangled. This lack of positivity can be witnessed by the so-called negativity [48, 49, 28]:

$$\mathcal{N}(\rho) = \frac{1}{2} \left(\|\tilde{\rho}\|_1 - \text{Tr}[\rho] \right), \quad \|\tilde{\rho}\|_1 = \text{Tr} \left[\sqrt{\tilde{\rho}^\dagger \tilde{\rho}} \right]. \quad (10)$$

which is nonvanishing only in presence of a non positive $\tilde{\rho}$. Although this criterion is not exhaustive (there are entangled states that remain positive under partial transposition), it results much more reliable in systems made of identical particles [11, 12]. Indeed, the operation of partial transposition gives a necessary and sufficient criteria for entanglement detection for very general classes of bosonic states (9), *e.g.* in presence of only two modes ($M = 2$), or, in the generic case of arbitrary M , when the $(\mathcal{A}_1, \mathcal{A}_2)$ -bipartition is such that the algebra \mathcal{A}_1 is generated by creation and annihilation operators of just one mode, while the remaining $M - 1$ modes generates \mathcal{A}_2 .

Even more interestingly, it turns out that entangled N -body bosonic states need to be of a definite, specific form [12]:

Proposition 1. *A generic $(m, M - m)$ -mode bipartite state (9) is entangled if and only if it can not be cast in the following block diagonal form*

$$\rho = \sum_{k=0}^N p_k \rho_k, \quad \sum_{k=0}^N p_k = 1, \quad \text{Tr}[\rho_k] = 1, \quad (11)$$

with

$$\rho_k = \sum_{\sigma, \sigma', \tau, \tau'} \rho_{k\sigma\sigma', k\tau\tau'} |k, \sigma; N - k, \sigma'\rangle \langle k, \tau; N - k, \tau'|, \quad \sum_{\sigma, \sigma'} \rho_{k\sigma\sigma', k\sigma\sigma'} = 1, \quad (12)$$

(*i.e.* at least one of its non-diagonal coefficients $\rho_{k\sigma\sigma', l\tau\tau'}$, $k \neq l$, is nonvanishing), or if it can, at least one of its diagonal blocks ρ_k is non-separable.⁵

Proof. Assume first that the state ρ can not be written in block diagonal form; one can then show [12] that it is not left positive by the operation of partial transposition and therefore it is entangled. Next, take ρ in block diagonal form as in (11), (12) above. If all its blocks ρ_k are separable, then clearly ρ itself results separable. Then, assume that at least one of the diagonal blocks is entangled. By mixing it with the remaining blocks as in (11) will not spoil its entanglement since all blocks ρ_k have support on orthogonal spaces; as a consequence, the state ρ results itself non-separable. \square

Having found the general form of non-separable N -boson states, one can next ask how robust is their entanglement content against mixture with other states. This question has been extensively studied for states of distinguishable particles [35, 36, 49, 50, 28]; in the next section we shall analyze to what extent the results obtained in that case can be extended to systems with a fixed number of bosons.

⁵For each block ρ_k , separability is understood with reference to the isomorphic structure $\mathbb{C}^{D_k} \otimes \mathbb{C}^{D_{N-k}}$ mentioned before (see *Remark 3*).

3 Robustness of entanglement

Several measures of entanglement have been introduced in the literature with the aim of characterizing quantum correlations and its usefulness in specific applications [28]-[30]. Starting with the entanglement of formation, most of these measures point to the quantification of the entanglement content of a given state. A different approach to this general problem has been proposed in [35, 36]: the idea is to obtain information about the magnitude of non-classical correlations contained in a state ρ by studying how much it can be mixed with other states before becoming separable.

More precisely, let us indicate with \mathcal{M} the set of all systems states and by $\mathcal{S} \subset \mathcal{M}$ that of separable ones; then, with reference to an arbitrary $(m, M - m)$ -mode bipartition, one can introduce the following definition:

Definition 4. *Given a state ρ , its **robustness of entanglement** is defined by*

$$R(\rho) = \inf \left\{ t \mid t \geq 0, \exists \sigma \in \mathcal{M}' \subset \mathcal{M}, \text{ for which } \eta \equiv \frac{\rho + t\sigma}{(1+t)} \in \mathcal{S} \right\}, \quad (13)$$

i.e. it is the smallest, non-negative t such that a state σ exists for which the (unnormalized) combination $\rho + t\sigma$ is separable.

Actually, various forms of robustness have been introduced: they all share the definition (13), but differ in the choice of the subset \mathcal{M}' from which the mixing state σ should be drawn. In particular, one talks of *generalized robustness* R_g when σ can be any state [36], while simply of *robustness* R_s when σ must be separable [35].

All robustness defined in this way satisfy nice properties; more specifically, they result entanglement monotones, *i.e.* they are invariant under local operations and classical communication, and convex, $R(\lambda\rho_1 + (1-\lambda)\rho_2) \leq \lambda R(\rho_1) + (1-\lambda)R(\rho_2)$; further, $R(\rho)$ in (13) is vanishing if and only if ρ itself is separable. Although implicitly proven in the case of states of distinguishable particles [35, 36], these properties hold true also in the case of N -boson systems. Nevertheless, there are striking differences in the behavior of the robustness of states of identical particles with respect to what is known in systems with distinguishable constituents.

Let us first focus on the robustness $R_s(\rho)$, that measures how strong is the entanglement content of a state ρ when mixed with separable states. One finds:

Proposition 2. *The robustness of entanglement of a generic $(m, M - m)$ -mode bipartite state ρ is given by*

$$R_s(\rho) = \sum_{k=0}^N p_k R_s(\rho_k), \quad (14)$$

for states that are in block diagonal form as in (11), (12), while it is infinitely large otherwise.

Proof. From the results of *Proposition 1*, we know that separable N -boson states must be block diagonal. If the state ρ is not in this form, it can never be made block diagonal by

mixing it with any separable one; therefore, in this case, the combination $\rho + t \sigma$ will never be separable, unless t is infinitely large.

Next, consider the case in which the state ρ is in block diagonal form, *i.e.* it can be written as in (11), (12). First, if ρ is separable, then clearly $R_s(\rho) = 0$. For an entangled ρ , one can discuss each block ρ_k separately: this is allowed since they have support on orthogonal Hilbert subspaces. Then, let us indicate by t_k the robustness of block density matrix ρ_k ; by *Remark 3* and the definition of robustness, the numbers t_k 's are finite and positive, vanishing only when the corresponding state ρ_k is separable [35]. More specifically, for each k , there exist separable states σ_k and η_k , such that:

$$\rho_k + t_k \sigma_k = (1 + t_k) \eta_k . \quad (15)$$

Multiplying both sides of this relation by the positive number p_k and then summing over k , one gets

$$\rho + t \sigma = (1 + t) \eta , \quad t = \sum_{k=0}^N p_k t_k , \quad (16)$$

where the separable states σ and η are explicitly given by

$$\sigma = \sum_{k=0}^N \left(\frac{p_k t_k}{t} \right) \sigma_k , \quad \eta = \sum_{k=0}^N \left(\frac{1 + t_k}{1 + t} \right) p_k \eta_k . \quad (17)$$

To prove that indeed t given in (16) is really the robustness of ρ , one needs to check that no better decomposition

$$\rho + t' \sigma' = (1 + t') \eta' , \quad (18)$$

with $t' \leq t$, exists. In order to show this, let proceed *ad absurdum* and assume that such decomposition can indeed be found. Since the states σ' and η' are separable, they must be block diagonal, *i.e.* of the form $\sigma' = \sum_k q_k \sigma'_k$ and $\eta' = \sum_k r_k \eta'_k$ with $\sum_k q_k = \sum_k r_k = 1$ and σ'_k and η'_k separable density matrices. By the orthogonality of the Hilbert subspaces with fixed k , from (18) one then gets

$$p_k \rho_k + t' q_k \sigma'_k = (1 + t') r_k \eta'_k , \quad (19)$$

and further, by taking its trace, $p_k + t' q_k = (1 + t') r_k$. In addition, from the previous identity, one sees that the combination

$$\rho_k + t' \frac{q_k}{p_k} \sigma'_k , \quad (20)$$

is separable. By definition of robustness of the block ρ_k as given in (15), it then follows that:

$$t' \frac{q_k}{p_k} \geq t_k , \quad (21)$$

ore equivalently, $t_k p_k \leq t' q_k$. By summing over k , one then finds: $t \leq t'$; this result is compatible with the initial assumption $t \geq t'$ only if t' coincides with t . Therefore, the

robustness of the block diagonal state ρ is indeed given by the weighted sum of the robustness of each block. \square

The problem of finding the robustness $R_s(\rho)$ of a generic N -boson state ρ is then reduced to the more manageable task of identifying the robustness of its diagonal blocks, which are finite-dimensional density matrices for which standard techniques and results can be used.

Remark 4: i) A remarkable property of the robustness of entanglement of states describing distinguishable particles is that it is equal to the negativity for pure states. In the case of identical particles, this property does not hold anymore as the robustness of entanglement of non-block diagonal pure states is infinitely large. Nevertheless, one can easily show that in general for pure states: $\mathcal{N}(\rho) \leq R_s(\rho)$.

ii) The robustness of entanglement of states that, with respect to the given partition, result mixtures of pure block states, $\rho = \sum_k p_k \rho_k$, $\rho_k^2 = \rho_k$, is equal to their negativity $R_s(\rho) = \sum_k p_k \mathcal{N}(\rho_k) = \mathcal{N}(\rho)$, since now $R_s(\rho_k) = \mathcal{N}(\rho_k)$, as in the standard case (see [12]). \square

More difficult appears the task of computing the generalized robustness $R_g(\rho)$ of a generic N -boson state: only upper bounds can in general be given. In any case, note that in general $R_g(\rho) \leq R_s(\rho)$ since the optimization procedure of *Definition 4* is performed over a larger subset of states in the case of the generalized robustness.

By fixing as before a $(m, M - m)$ -mode bipartition, a first bound on $R_g(\rho)$ can be easily obtained. Let us extract from ρ its diagonal part ρ_D , as defined in terms of a Fock basis determined by the given bipartition (*cf.* (6)), and call $\rho_{ND} \equiv \rho - \rho_D$ the rest. By definition of separability, ρ_D is surely separable with respect to the chosen bipartition, so that an easy way to get a separable state by mixing ρ with another state σ is to subtract from it its non-diagonal part. However, $-\rho_{ND}$ alone is not in general a density matrix since it might have negative eigenvalues. Let us denote by λ the modulus of its largest negative eigenvalue; then the quantity $\lambda \mathbb{1} - \rho_{ND}$, where $\mathbb{1}$ is the identity matrix, will surely be positive and therefore, once normalized, can play the role of the density matrix σ in the separable combination $\rho + t \sigma \equiv \rho_D + \lambda \mathbb{1}$. By taking the trace, one finds for the normalization factor t the following expression: $t = \lambda D$, where D is as before the dimension of the total Hilbert space. By definition of robustness, it then follows that

$$R_g(\rho) \leq \lambda D ; \quad (22)$$

as a consequence, the generalized robustness of a generic N -boson state is always finite.

A different bound on $R_g(\rho)$ can be obtained using a refined decomposition for ρ ,

$$\rho = \rho_B + \rho_{NB} , \quad \rho_B = \sum_{k=0}^N p_k \rho_k , \quad \sum_{k=0}^N p_k = 1 , \quad \text{Tr}[\rho_k] = 1 , \quad (23)$$

where ρ_B is the block diagonal part of ρ , whose blocks ρ_k can be written as in (12), while $\rho_{NB} \equiv \rho - \rho_B$ is the rest, containing the non block diagonal pieces. One can first ask for the

generalized robustness of ρ_B , which is a *bona fide* state, being a normalized, positive matrix.⁶ Quite in general, one has:

Proposition 3. *The generalized robustness of entanglement of a generic $(m, M - m)$ -mode bipartite state ρ given in block diagonal form as in (11), (12) is given by*

$$R_g(\rho) = \sum_{k=0}^N p_k R_g(\rho_k) . \quad (24)$$

Proof. It is the same as in *Proposition 2*; the only difference is that now the states σ_k are in general entangled. \square

As a consequence, for a generic state as in (23) above, due to the presence of the additional term ρ_{NB} , one surely has: $R_g(\rho) \geq R_g(\rho_B) = \sum_{k=0}^N p_k R_g(\rho_k)$.

To get an upper bound for $R_g(\rho)$, let us consider the form that the (unnormalized) density matrices σ must take in order to make the combination $\rho + \sigma$ separable; by *Definition 4*, the generalized robustness of entanglement coincides with minimum value of their traces: $R_g(\rho) = \inf\{\text{Tr}[\sigma]\}$. Since the combination $\rho + \sigma$ is separable, it must be in block diagonal form; therefore, σ must surely contain the contribution $-\rho_{NB}$. Further, it must also take care of the entanglement in the remaining block diagonal term ρ_B ; in view of the results of *Proposition 3*, this can be obtained (and in an optimal way) by the contribution $\sum_{k=0}^N p_k R_g(\rho_k) \sigma_k$, where σ_k is the optimal density matrix that makes the diagonal block ρ_k separable.

However, while $\sum_{k=0}^N p_k R_g(\rho_k) \sigma_k$ is a positive matrix, in general ρ_{NB} is not; therefore σ should contain a further contribution, a (unnormalized) positive and separable matrix $\tilde{\sigma}$ curing the negativity induced by ρ_{NB} . As a consequence, the generic form of the positive matrix σ making the combination $\rho + \sigma$ separable is given by:

$$\sigma = \sum_{k=0}^N p_k R_g(\rho_k) \sigma_k - \rho_{NB} + \tilde{\sigma} , \quad (25)$$

and the computation of the generalized robustness of ρ is reduced to the determination of the optimal $\tilde{\sigma}$; indeed:

$$R_g(\rho) = \sum_{k=0}^N p_k R_g(\rho_k) + \inf\{\text{Tr}[\tilde{\sigma}]\} . \quad (26)$$

Upper bounds on $R_g(\rho)$ can then be obtained by estimating the above minimum value, through specific choices of $\tilde{\sigma}$.

A simple possibility for curing the non positivity of $-\rho_{NB}$ is to add to it a matrix proportional to the modulus of its largest negative eigenvalue; in general, the value of this eigenvalue is however difficult to estimate. Another possibility is suggested by the general

⁶This is a direct consequence of the positivity of ρ , since ρ_B is made of its principal minors.

theory of positive matrices (see Theorem 6.1.1 and 6.1.10 in [51]): a sufficient condition for a generic hermitian matrix M_{ij} to be positive is that it must be “diagonally dominated”, *i.e.* $M_{ii} \geq \sum_{j \neq i} |M_{ij}|$, $\forall i$.⁷ Then, in a fixed separable basis, by choosing for $\tilde{\sigma}$ the diagonal matrix whose entries are given by the sum of the modulus of the elements of the corresponding rows of $-\rho_{NB}$, the matrix σ in (25) results positive and makes the combination $\rho + \sigma$ separable. One can then conclude that:

$$R_g(\rho) \leq \sum_{k=0}^N p_k R_g(\rho_k) + \|\rho_{NB}\|_{\ell_1} , \quad (27)$$

where, for any matrix M , $\|M\|_{\ell_1} = \sum_{i,j} |M_{ij}|$ is the so-called ℓ_1 -norm (see [51]).⁸

In presence of just two modes, $M = 2$, each of which forming a partition, the above considerations further simplify. In this case, the Fock basis in (6) is given by the set of $N + 1$ vectors $\{|k; N - k\rangle, 0 \leq k \leq N\}$, without the need of further labels; indeed, the N bosons can occupy either one of the two modes. Notice that by (4) this set of Fock vectors constitutes the only basis made of separable pure states [12]. In this basis, a generic density matrix for the system can then be written as:

$$\rho = \sum_{k,l=0}^N \rho_{kl} |k; N - k\rangle \langle l; N - l| , \quad \sum_{k=0}^N \rho_{kk} = 1 . \quad (28)$$

By *Proposition 1*, once adapted to this simplified case, it follows that a state as in (28) is separable if and only if $\rho_{kl} \sim \delta_{kl}$, *i.e.* the density matrix ρ is diagonal in the Fock basis. As a consequence, an entangled state can never be made separable by mixing it with a separable state, so that its robustness of entanglement results always infinite.

In the case of the generalized robustness, since there are only diagonal and off-diagonal terms and no blocks, the above discussed upper bounds (22) and (27) simplify, becoming:

$$i) R_g(\rho) \leq \lambda(N + 1) , \quad ii) R_g(\rho) \leq \|\rho_{ND}\|_{\ell_1} , \quad (29)$$

where, as before, ρ_{ND} is the non diagonal part of ρ in the Fock basis, and λ the modulus of its largest positive eigenvalue. These bounds can be explicitly evaluated for specific classes of states as shown below.

Examples: *i)* Let us first consider pure states of the form:

$$\rho \equiv |\psi\rangle \langle \psi| , \quad |\psi\rangle = \frac{1}{\sqrt{N+1}} \sum_{k=0}^N p_k |k; N - k\rangle , \quad p_k = e^{i\varphi_k} . \quad (30)$$

⁷Note that this condition is base-dependent: nonequivalent conditions are obtained by expressing the matrix M in different basis.

⁸The same procedure can also be applied to the previously used decomposition of ρ into its diagonal and off-diagonal parts: $\rho = \rho_D + \rho_{ND}$; the mixing matrix σ would now be composed by $-\rho_{ND}$ plus a diagonal matrix whose entries are given by the sums of the modulus of the elements of each row of ρ_{ND} . In this case, one easily finds that: $R_g(\rho) \leq \|\rho_{ND}\|_{\ell_1}$; although in general $\|\rho_{ND}\|_{\ell_1} \geq \|\rho_{NB}\|_{\ell_1}$, this constitutes a different upper bound for the generalized robustness, independent from that given in (27).

The non-diagonal part of the matrix ρ is given by $\rho_{ND} = (\Phi - \mathbb{1})/(N + 1)$ where $\Phi = \sum_{k,l} e^{i(\varphi_k - \varphi_l)} |k; N - k\rangle\langle l; N - l|$ is the $(N + 1) \times (N + 1)$ matrix of phase differences; its eigenvalues are zero and $N + 1$, so that the largest negative eigenvalue of $-\rho_{ND}$ is in modulus $N/(N + 1)$. From the first bound in (29) one therefore gets: $R_g(\rho) \leq N$. This is also the result of the second bound, since the norm $\|\rho_{ND}\|_{\ell_1}$ is also equal to N .

ii) Nevertheless, the two bounds in (29) do not in general coincide, as can be seen by slightly generalizing the states in (30) by allowing the coefficients p_k to acquire a non unit norm, $p_k = |p_k|e^{i\varphi_k}$, $\sum_k |p_k|^2 = 1$. By choosing the norms and phases of the p_k 's to be uniformly distributed, one can easily generate states for which the second bound in (29) is more stringent.

iii) Note, however, that the hierarchy of the bounds in (29) can be reversed, as it happens for instance with the following mixed state:

$$\rho = \frac{1}{N+1} \sum_{k=0}^N |k; N-k\rangle\langle k; N-k| - \frac{1}{N(N+1)} \sum_{\substack{k,l=0 \\ k \neq l}}^N |k; N-k\rangle\langle l; N-l|. \quad (31)$$

Indeed, now one has $\rho_{ND} = (\mathbb{1} - E)/N(N + 1)$, where E is the matrix with all entries equal to one. One easily checks that $\|\rho_{ND}\|_{\ell_1} = 1$, while the modulus of the largest negative eigenvalue λ of $-\rho_{ND}$ is $1/N(N + 1)$. Therefore, from the first bound in (29) one gets $R_g(\rho) \leq 1/N$, which is lower than the second one. \square

4 On the geometry of N -boson states

As shown by the previous results, the properties of the states of a system of N -bosons result rather different and to a certain extent richer than those describing distinguishable particles. This is surely a consequence of the indistinguishability of the system elementary constituents, but also of the presence of the additional constraint that fixes the total number of particles to be N . This additional “rigidity” allows nevertheless a detailed description of the geometric structure of the set of N -boson states.

Let us first consider the set of entangled states. As mentioned before, for systems of identical particles, negativity, as defined in (10), results a much more exhaustive entanglement criteria than in systems of distinguishable particles: it seems then appropriate to look for states that maximize it.

Proposition 4. *Given any bipartition $(m, M - m)$, the negativity $\mathcal{N}(\rho)$ is maximized by pure states that have all the Schmidt coefficients nonvanishing and equal to a normalizing constant.*

Proof. First observe that the negativity is a convex function [49], *i.e.* it satisfies the inequality

$$\mathcal{N}\left(\sum_i p_i \rho_i\right) \leq \sum_i p_i \mathcal{N}(\rho_i), \quad p_i \geq 0, \quad \sum_i p_i = 1, \quad (32)$$

for any convex combination $\sum_i p_i \rho_i$ of density matrices ρ_i . Since any density matrix can be written as a convex combination of projectors over pure states, one can limit the search to pure states; in a given $(m, M - m)$ -bipartition, they can be expanded in terms of the Fock basis (6) as:

$$|\psi\rangle = \sum_{k=0}^N |\psi_k\rangle, \quad |\psi_k\rangle \equiv \sum_{\sigma=1}^{D_k} \sum_{\sigma'=1}^{D_{N-k}} \psi_{k\sigma\sigma'} |k, \sigma; N - k, \sigma'\rangle, \quad \sum_k \sum_{\sigma\sigma'} |\psi_{k\sigma\sigma'}|^2 = 1. \quad (33)$$

As observed before, for each k , the set of vectors $\{|k, \sigma; N - k, \sigma'\rangle\}$ span a subspace $\mathcal{H}_k \subset \mathcal{H}$ of finite dimension $D_k D_{N-k}$ of the total Hilbert space \mathcal{H} , and the component $|\psi_k\rangle$ is a vector of this space. Recalling *Remark 3*, one can then write it in Schmidt form,

$$|\psi_k\rangle \equiv \sum_{\alpha=1}^{\mathcal{D}_k} \tilde{\psi}_{k\alpha} ||k, \alpha; N - k, \alpha\rangle; \quad (34)$$

in this decomposition, the orthonormal vectors $\{||k, \alpha; N - k, \alpha\rangle\}$ form the Schmidt basis, with $\mathcal{D}_k = \min\{D_k, D_{N-k}\}$ (*cf.* (6)), while the Schmidt coefficients $\tilde{\psi}_{k\alpha}$ are non negative real numbers, satisfying the normalization condition $\sum_k \sum_{\alpha} (\tilde{\psi}_{k\alpha})^2 = 1$. In this new basis, one can easily compute the negativity of the density matrix $|\psi\rangle\langle\psi|$,

$$\mathcal{N}(|\psi\rangle\langle\psi|) = \frac{1}{2} \left[\left(\sum_{k=0}^N \sum_{\alpha=1}^{\mathcal{D}_k} \tilde{\psi}_{k\alpha} \right)^2 - 1 \right]. \quad (35)$$

Clearly, the negativity increases monotonically with the sum $\sum_k \sum_{\alpha} \tilde{\psi}_{k\alpha}$, that therefore needs to be maximized under the constraint $\sum_k \sum_{\alpha} (\tilde{\psi}_{k\alpha})^2 = 1$. One easily shows that the maximum is obtained when all coefficients $\tilde{\psi}_{k\alpha}$ are equal and constant,

$$\tilde{\psi}_{k\alpha} = \frac{1}{\sqrt{\mathcal{D}}}, \quad \mathcal{D} = \sum_{k=0}^N \mathcal{D}_k, \quad (36)$$

so that, $\mathcal{N}(|\psi\rangle\langle\psi|) = (\mathcal{D} - 1)/2$. Further, all Schmidt coefficients need to be non vanishing in order to get this maximum value for \mathcal{N} ; indeed, for any state $|\psi'\rangle$ with Schimdt number $\mathcal{D}' < \mathcal{D}$, one has: $\mathcal{N}(|\psi'\rangle\langle\psi'|) < (\mathcal{D} - 1)/2$. \square

In analogy with the standard case, states for which the negativity reaches the maximum value $(\mathcal{D} - 1)/2$ can be called “maximally entangled states”.

Remark 5: i) Given a fixed bipartition $(m, M - m)$, let us consider such a maximally entangled state $\rho = |\psi\rangle\langle\psi|$. By tracing over the degrees of freedom of the modes pertaining to the second partition, one obtains the reduced density matrix $\rho^{(1)}$ describing the first m modes only:

$$\rho^{(1)} = \frac{1}{\mathcal{D}} \sum_{k=0}^N \sum_{\alpha=1}^{\mathcal{D}_k} ||k, \alpha\rangle \langle\langle k, \alpha|; \quad (37)$$

similarly, by tracing over the first partition, one obtains the reduced density matrix $\rho^{(2)}$ describing the second $N - k$ modes. In the case of distinguishable particles, either $\rho^{(1)}$ or $\rho^{(2)}$ is proportional to the identity matrix; this is no longer true here: in fact, given a block with k fixed, this happens only when $D_k \leq D_{N-k}$, *i.e.* when $\mathcal{D}_k = D_k$.

ii) On the other hand, given the expression (37), one easily computes its von Neumann entropy, obtaining: $S(\rho^{(1)}) = -\text{Tr}[\rho^{(1)} \ln \rho^{(1)}] = \ln \mathcal{D}$, as in the standard case.

iii) Similarly, the purity of $\rho^{(1)}$ is given by: $\text{Tr}[(\rho^{(1)})^2] = 1/\mathcal{D}$. This result and that of the previous remark can equally be taken as alternative, equivalent definitions of the notion of maximally entangled states. \square

Let us now come to the analysis of the space \mathcal{S} of separable states as determined by the generic bipartition $(m, M - m)$. Among them, the totally mixed state ρ_{mix} , proportional to the unit matrix, stands out because of its peculiar properties. In fact, recall that in the case of distinguishable particles, ρ_{mix} always lays in the interior of \mathcal{S} [48, 29]; instead, now one finds:

Proposition 5. *Given any bipartition $(m, M - m)$ and an associated separable basis made of the Fock vectors $|k, \sigma; N - k, \sigma'\rangle$, the totally mixed state,*

$$\rho_{\text{mix}} = \frac{1}{D} \sum_{k=0}^N \sum_{\sigma, \sigma'} |k, \sigma; N - k, \sigma'\rangle \langle k, \sigma; N - k, \sigma'|, \quad (38)$$

lays on the border of the space \mathcal{S} of separable states.

Proof. Let us take a state ρ_{ent} which can not be written in block diagonal form as in (11), (12); by *Proposition 1*, it is entangled. Then, consider the combination

$$\rho_{\epsilon} = \frac{1}{1 + \epsilon} (\rho_{\text{mix}} + \epsilon \rho_{\text{ent}}), \quad \epsilon > 0. \quad (39)$$

For any ϵ , this combination will never be separable, since it is not block diagonal. In other terms, in the vicinity of ρ_{mix} , one always find entangled states, so that it must lay on the border of \mathcal{S} . \square

Remark 6: i) Note that similar considerations apply to all separable states: there always exist small perturbations of separable, necessarily block diagonal, states that make them not block diagonal, hence entangled. Instead, in the case of distinguishable particles, almost all separable states remain separable under sufficiently small arbitrary perturbations [29, 52].

ii) Further, using analogous steps, one can show that a Werner-like state, $\rho_W = p |\psi\rangle \langle \psi| + (1 - p) \rho_{\text{mix}}$, $0 \leq p \leq 1$, with $|\psi\rangle$ a maximally entangled state, is entangled for any nonvanishing value of the parameter p , while for distinguishable d -level particles, this happens only when $1/(d + 1) < p \leq 1$ [53]. \square

The result of *Proposition 5* is most strikingly illustrated by considering a system of N bosons that can occupy two modes ($M = 2$), each of which forming a partition. In the

previous section, we have seen that a generic state ρ for this system can be written in terms of the Fock basis as in (28); further, it results separable if and only if the density matrix ρ is diagonal in this basis. Instead, ρ_ϵ given in (39) will develop non-diagonal entries as long as ϵ starts to be non vanishing.

As mentioned before, in this case the set of $N+1$ vectors $\{|k; N-k\rangle\}$ constitutes the only basis made of separable pure states [12]. Therefore, the decomposition of a generic separable state ρ in terms of projections on separable states turns out to be unique. As a consequence, the set \mathcal{S} of separable states is a sub-variety of the convex space \mathcal{M} of all states that turns out to be a simplex, whose vertices are given precisely by these projections.

The space of states of N bosons confined in two modes is then much more geometrically structured than in the case of systems made of distinguishable particles. Indeed, given a complete set of observables $\{\mathcal{O}_i\}$, one can decompose any density matrix ρ as:

$$\rho = \sum_i \rho_i \left(\frac{\sqrt{\rho} \mathcal{O}_i \sqrt{\rho}}{\text{Tr}[\mathcal{O}_i \rho]} \right), \quad \rho_i \equiv \text{Tr}[\mathcal{O}_i \rho], \quad \mathcal{O}_i > 0, \quad \sum_i \mathcal{O}_i = 1; \quad (40)$$

this decomposition is over pure states, whenever the operators \mathcal{O}_i are chosen to be projectors. Therefore, in general, there are infinite ways of expressing a density matrix as a convex combination of projectors, even when these projectors are made of separable states. As seen, this conclusion no longer holds for systems of identical particles.

The totally mixed state ρ_{mix} has also another interesting property:⁹

Proposition 6. *The totally mixed state is the only state that remains separable for any choice of bipartition.*

Proof. First, let us again consider the simplest case $M=2$. Any Bogolubov transformation maps the set of creation and annihilation operators a_i^\dagger and a_i , $i=1,2$ into new ones b_i^\dagger , b_i , $i=1,2$; a simple example is given by

$$b_1 = \frac{a_1 + a_2}{\sqrt{2}}, \quad b_2 = \frac{a_1 - a_2}{\sqrt{2}}, \quad (41)$$

and their hermitian conjugates. The operators b_i^\dagger , b_i define a new bipartition $(\mathcal{B}_1, \mathcal{B}_2)$ of the full algebra \mathcal{B} of bounded operators, distinct from original one $(\mathcal{A}_1, \mathcal{A}_2)$ generated by a_i^\dagger , a_i . States (and operators as well) that are local in one bipartition might turn out to be non-local in the other. For instance, the Fock states $\{|k, N-k\rangle\}$ result entangled with respect to the new bipartition defined by the transformation (41); indeed, one finds:

$$|k, N-k\rangle = \frac{1}{2^{N/2}} \frac{1}{\sqrt{k!(N-k)!}} \sum_{r=0}^k \sum_{s=0}^{N-k} \binom{k}{r} \binom{N-k}{s} (-1)^{N-k-s} (b_1^\dagger)^{r+s} (b_2^\dagger)^{N-r-s} |0\rangle, \quad (42)$$

⁹For systems of distinguishable particles, the problem of finding so-called “absolutely separable states”, *i.e.* states that are separable in any choice of tensor product structure, has been discussed in [54, 29].

so that $|k, N - k\rangle$ is a combination of $(\mathcal{B}_1, \mathcal{B}_2)$ -separable states. A similar conclusion applies to the mixed states (28): in general, any separable state $\rho_{\text{sep}} = \sum_k \rho_k |k; N - k\rangle\langle k; N - k|$ is mapped by a Bogolubov transformation into a non-diagonal density matrix, and therefore into an entangled state. In fact, one can always find a unitary transformation U that maps any diagonal matrix ρ into a non-diagonal one $U\rho U^\dagger$. In particular, when ρ is not degenerate, the transformed matrix $U\rho U^\dagger$ results diagonal only if the operator U is itself a diagonal matrix; in this case however the corresponding Bogolubov transformation results trivial and does not define a new partition. The only density matrix that remains invariant under all unitary transformations is the one proportional to the unit matrix, *i.e.* the totally mixed one. This conclusion can easily be extended to the multimode case: given any separable state in a given $(m, M - m)$ bipartition, $\rho_{\text{sep}} = \sum_{k=0}^N \sum_{\sigma, \sigma'} \rho_{k\sigma\sigma'} |k, \sigma; N - k, \sigma'\rangle\langle k, \sigma; N - k, \sigma'|$, one can always construct a Bogolubov transformation that maps it into an entangled one: it is sufficient to apply the above considerations to any couple of modes belonging to separate partitions. Therefore, also in this more general setting, only the state ρ_{mix} in (38) is left invariant by all Bogolubov transformations. \square

Thanks to these results, the global geometry of the space of N -boson states starts to emerge more clearly. Again the two-mode case is easier to describe. We have seen that by fixing a bipartition one selects the sets \mathcal{S} of separable states; these form a sub-variety of the convex space \mathcal{M} of all states, forming a $(N + 1)$ -dimensional simplex, with the projectors over Fock states as generators. Changing the bipartition through a Bogolubov transformation produces a new simplex, having only one point in common with the starting one, the state ρ_{mix} . The geometry of the space of two-mode N -boson states has therefore a sort of star-like topology, with the various simplexes sharing just one point, the totally mixed state.

The case of M modes is more complex. For a fixed bipartition, the space of separable states is a sub-space of the convex space of all states which is not any more strictly a simplex: the decomposition of generic separable state ρ is no longer unique, since the Fock states in (6) are no longer the only separable pure states: for each k , reshufflings over the indices σ, σ' are still allowed. Nevertheless, also in this case the global state space presents a sort of star-like topology: only one point is shared by all separable bipartition sub-spaces, the totally mixed state ρ_{mix} .

5 Outlook

In many-body systems composed by a fixed number of identical particles, the associated Hilbert space no longer exhibits the familiar particle tensor product structure. The usual notions of separability and entanglement based on such structure is therefore inapplicable: a generalized definition of separability is needed and can be given in terms of commuting algebras of observables instead of focusing on the particle tensor decomposition of states. The selection of these algebras is largely arbitrary, making it apparent that in systems of identical particles entanglement is not an absolute notion: it is strictly bounded to specific choices of commuting sets of system observables.

Using these generalized definitions, we have studied bipartite entanglement in systems composed by N identical bosons that can occupy M different modes. More specifically, we have analyzed to what extent entangled states result robust against mixing with another state, either separable or entangled. We have found that in general, the entanglement contained in bosonic states is much more robust than the one found in systems of distinguishable particles. This result has been obtained by analyzing the so-called *robustness* and *generalized robustness* of entanglement, of which explicit expressions and upper bounds have been respectively given. A quite general characterization of the geometry of the space of bosonic states has also been obtained: this space exhibits a star-like structure composed by intersecting subspaces each of one determined by a given bipartition through the subset of separable states. All these separable subspaces share one and only one point, the totally mixed state, hence the star-shape topology.

As a final remark, notice that all above results can be generalized to the case of systems where the total number of particles is not fixed, but commutes with all physical observables (*i.e.* we are in presence of a superselection rule [55]). In such a situation, a general density matrix ρ can be written as an incoherent mixture of states ρ_N with fixed number N of particles, having the general form (9); explicitly:

$$\rho = \sum_N \lambda_N \rho_N, \quad \lambda_N \geq 0, \quad \sum_N \lambda_N = 1. \quad (43)$$

The state ρ is thus a convex combination of matrices ρ_N having support on orthogonal spaces. Arguments similar to those used in proving *Proposition 2* (and *Proposition 3*) allow us to conclude that both the robustness and the generalized robustness of entanglement of the state ρ in (43) is the weighted average of the robustness of the components ρ_N , *i.e.* $R(\rho) = \sum_N \lambda_N R(\rho_N)$ for both cases. The problem of computing the robustness of incoherent particle number mixtures (43) is then reduced to that of determining the robustness of the corresponding components at fixed particle number, for which the considerations and results discussed in the previous sections apply.

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